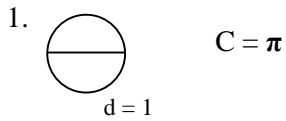


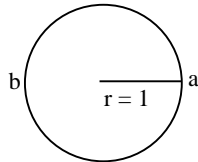
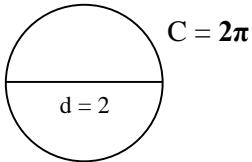
Calculating Pi Using Trigonometry and Polygons Answers Name: LIACOS EDUCATIONAL MEDIA

Pi (or π) is the ratio of a circle's circumference (C) to its diameter (d). Mathematically, we can write this as a formula: $\pi = C/d$. By now most of you know that π is approximately 3.142 or $\frac{22}{7}$.

A circle with a diameter of 1 unit therefore has a circumference of π , or 3.142 units.



2. What is the circumference of the following circle?



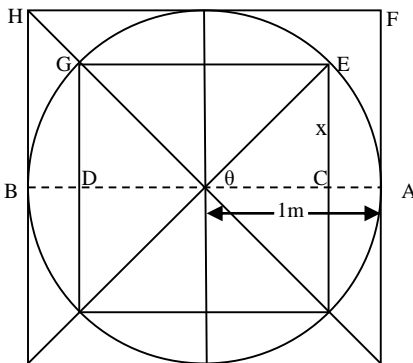
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3. What is the circumference of the **semi-circle** in question 2? π

4. So, if the radius is one unit, the length of the semi-circle (from a to b) is π .

But how did humans find out what the actual value of π was? The famous Greek mathematician Archimedes, who was born in about 287 BC, was one of the first people to devise a method of estimating π . He knew he could place a polygon on the inside of a circle and place another polygon on the outside of the circle. Polygons have straight lines, so we can get an idea of their perimeters more easily. The circumference of the circle must lie somewhere between the perimeters of the two polygons. If we can estimate the circumference, we can estimate π . **In this activity, you are going to use your knowledge of trigonometry to calculate π .** Let's start by using squares.

Square: A 4-sided polygon.



Remember, the radius of the circle is 1metre.

What is the distance from A to B around the circle? It must be π . But what is π ? The length must be somewhere between the distance A to B along the outer square and the distance C to D along the inner square. This is how Archimedes started.

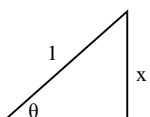
Outer square:

What is the distance around the outside square (from A to F to H to B)? **4m**

π must be less than this value so we write $\pi < 4$

Inner Square:

What is the distance around the inner square(from C to E to G to D)? We need trigonometry for this.



$\sin \theta = \text{opp/hyp} = x/1$

What angle is θ ? **45°**

What is the value of x? **0.707**

What is the distance around the inner square(from C to E to G to D)? **2.828**

π must be more than this value so we write $\pi > 2.828$

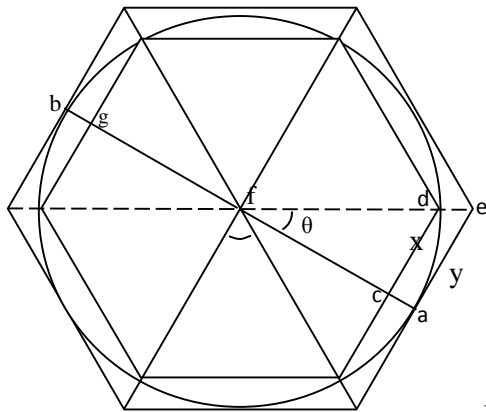
Putting the two together we can write

$2.828 < \pi < 4$

This is not very accurate! So let's do a polygon with more sides.

We could, of course take an average, and this will get us closer to Pi. However, Archimedes probably wasn't interested in estimating π . He wanted to use mathematics to be able to say things with certainty. At the end of his calculations, he could say with certainty that π was between $3\frac{1}{7}$ and $3\frac{10}{71}$.

Hexagon: A six-sided polygon.



Remember, the radius of the circle is 1metre. $fa = 1m$, $fd = 1m$

What is the distance from a to b around the semi-circle? π again. But what's π ?

What is the angle θ shown? $180/6 = 30^\circ$

Outer hexagon:

What is the distance y ? Trace the triangle aef in blue. To calculate “ y ” you will need “ \tan ”. $\tan \theta = \frac{opp}{adj} = \frac{y}{1} \therefore$

$$y = \tan \theta$$

$$y = \tan 30^\circ = 0.577$$

Therefore what is the distance from a to b around the outer hexagon?

$$0.577 \times 6 = 3.464$$

Therefore $\pi < 3.464$

Inner Hexagon:

What is the distance x ? Trace the triangle cdf in red. To calculate “ x ” you will need “ \sin ” again.

$$x = \sin \theta = \sin 30^\circ = 0.5$$

Therefore what is the distance from c to g around the inner hexagon?

$$0.5 \times 6 = 3$$

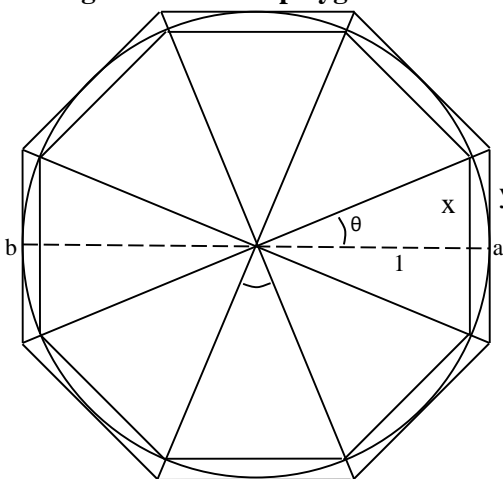
Therefore $\pi > 3$

Putting the two together we can write

$$3 < \pi < 3.464$$

This is a lot more accurate, but we can surely do better.

Octagon: an 8-sided polygon



What is the angle θ ? $180/8 = 22.5^\circ$

Outer Octagon:

What is the value of y ? $y = \tan \theta = \tan 22.5^\circ = 0.414$

What is the distance from a to b around the outer hexagon?

$$\text{distance}_{ab} = 0.414 \times 8 = 3.3137$$

Inner Octagon:

What is the value of x ?

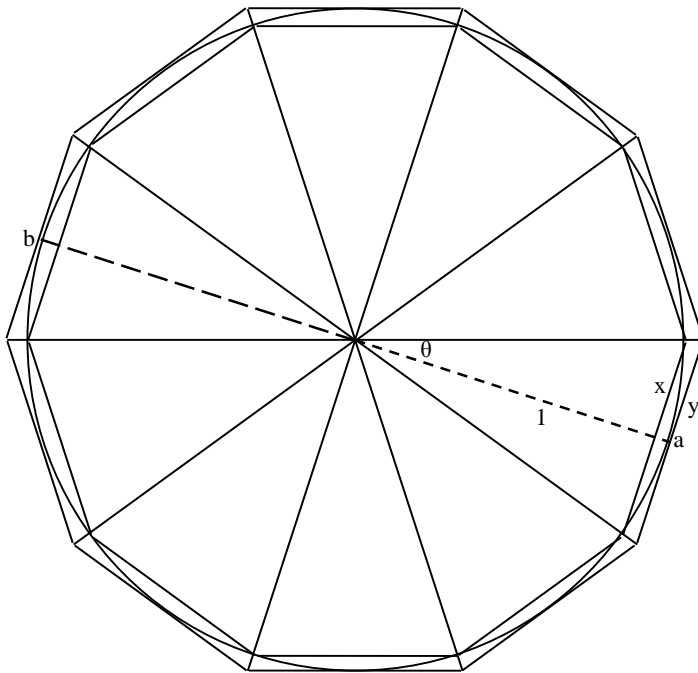
$$x = \sin 22.5^\circ = 0.383$$

What is the distance half way around the inner hexagon?

$$0.383 \times 8 = 3.061$$

Therefore $3.061 < \pi < 3.3137$

Decagon: a 10-sided polygon



What is the angle θ ? $180/10 = 18^\circ$

Outer Decagon:

What is the value of y ? $y = \tan\theta = \tan 18^\circ = 0.3249$

What is the distance from a to b around the outer decagon?

$$0.3249 \times 10 = 3.249$$

Inner Decagon:

What is the value of x ?

$$x = \sin\theta = \sin 18^\circ = 0.309$$

What is the distance half way around the inner decagon?

$$0.309 \times 10 = 3.09$$

Therefore $3.09 < \pi < 3.249$

Summarising...

Is there a pattern here? Of course there is! But how can we express the pattern mathematically. Remember, Maths is just like a language.

What if we had not a 4-sided polygon (a square) or a 10-sided polygon (a decagon) or any other specific number, but an n -sided polygon. To answer the following questions you will need to refer to your work above.

n-gon: an n-sided polygon.

What is the angle θ ? $180/n^\circ$

Outer n-gon:

What is the value of y ? $y = \tan(180/n)^\circ$

What is the distance from a to b around the outer n-gon?

$$\tan(180/n)^\circ \times n \quad \text{or, more neatly,} \quad n \tan(180/n)^\circ$$

Inner n-gon:

What is the value of x ?

$$x = \sin(180/n)^\circ$$

What is the distance from a to b around the inner n-gon?

$$\sin(180/n)^\circ \times n \quad \text{or, more neatly,} \quad n \sin(180/n)^\circ$$

Now for a formula

Therefore:
$$n \sin\left(\frac{180^\circ}{n}\right) < \pi < n \tan\left(\frac{180^\circ}{n}\right)$$

We now know that π is a number that we can't express exactly using decimals: $\pi = 3.141592653589793$ etc. etc. The decimals go on forever and never repeat. But by using various mathematical techniques like the one you have just used, we can get very accurate values for π . Let's now test the formula.

Questions

1. Using as many decimal points as your calculator allows, try finding π (which is 3.1415926535897932384626433832795...) for a value of

(a) $n=100$

(i) inner:

$$100 \sin (180/100)^{\circ} = 3.1410759$$

accurate to 3 decimal places

(ii) outer:

$$100 \tan (180/100)^{\circ} = 3.1426266$$

accurate to 2 decimal places

(b) $n=1000$

(i) inner:

$$1000 \sin (180/1000)^{\circ} = 3.14158748587956$$

accurate to 4 decimal places (depending on the way you round it)

(ii) outer:

$$1000 \tan (180/1000)^{\circ} = 3.14160$$

accurate to 3 decimal places (depending on rounding)

(c) $n=100,000$

(i) inner:

$$100000 \sin (180/100000)^{\circ} = 3.141592653073$$

accurate to 9 decimal places

(ii) outer:

$$100000 \tan (180/100000)^{\circ} = 3.1415926546233$$

accurate to 8 decimal places

2. How accurate is your formula?

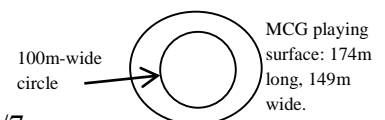
As we increased the number of sides on the polygons, they became closer and closer towards being circles. The accuracy of π therefore got more and more accurate.

3. Now Archimedes didn't have the use of sin or tan, but using a similar method to what we have used, and using a polygon of 96 sides, he calculated that $3 \frac{10}{71} < \pi < 3 \frac{10}{70}$. Re-write the inequation with decimals and state how accurate it is?

$$3.14084 < \pi < 3.142857 \quad \text{accurate to about 2 decimal places}$$

4. In about 480AD, the Chinese mathematician Zu Chongzhi found that the fraction $355/113$ was a close approximation of π . Convert this fraction to a decimal and state how accurate it is as an approximation to π . (This value was the most accurate approximation of π for the next 900 years)

$$355/113 = 3.14159292035 \quad \text{accurate to about 6 decimal places}$$



5. Calculate the circumference of a circle of 100m diameter using $\pi = 22/7$

$$C = \pi d = 22/7 \times 100 = 314.2857\text{m}$$

6. Calculate the circumference of a circle of 100m diameter using $\pi = 3.1415926$ (7 decimal places)

$$C = \pi d = 3.1415926 \times 100 = 314.1593\text{m}$$

7. Calculate the difference in your answers to questions 5 and 6 in centimetres.

$$314.2857\text{m} - 314.1593\text{m} = 0.1264\text{m} \approx 13\text{cm}$$

FINAL NOTE: According to wikipedia, a value of π truncated to 11 decimal places is accurate enough to calculate the circumference of a circle the size of the earth with a precision of a millimeter, and one truncated to 39 decimal places is sufficient to compute the circumference of any circle that fits in the observable universe to a precision comparable to the size of a hydrogen atom. So 3 or 4 decimal places is usually enough for most jobs!